TOTALLY GEODESIC SURFACES WITH ARBITRARILY MANY COMPRESSIONS

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ABSTRACT. A closed totally geodesic surface in the figure eight knot complement remains incompressible in all but finitely many Dehn fillings. In this paper, we show that there is no universal upper bound on the number of such fillings, independent of the surface. This answers a question of Ying-Qing Wu.

1. Introduction

Let M be a compact, connected, irreducible, orientable 3-manifold with torus boundary ∂M. A slope on ∂M is an isotopy class of simple closed curves on ∂M. We use $\Delta(\alpha, \beta)$ to denote the absolute value of the algebraic intersection number between the slopes α and β. It is shown in [4] that if F is a closed, orientable, embedded, incompressible surface in M with no incompressible annulus joining F and ∂M, and F compresses in the Dehn fillings $M(\alpha)$ and $M(\beta)$, then $\Delta(\alpha, \beta) \leq 2$. In [16], Wu improved this to $\Delta(\alpha, \beta) \leq 1$, and hence F remains incompressible in $M(\gamma)$ for all but at most three slopes γ.

If one drops the assumption that F be embedded, the previous theorem is not true; see [6]. However, for hyperbolic M such a surface F can compress in at most finitely many Dehn fillings $M(\gamma)$; see [1]. In fact in [18], it is shown that there is a bound on the number of fillings in which F can compress depending only on the genus of F, and not on the manifold M. Wu has asked whether there is any universal bound, independent of F, for this number (Question 6.6 in [17]). In this paper, we prove that no such universal bound exists. More precisely we prove

Theorem 1.1. *There exists a compact, connected, orientable, 3-manifold M, with torus boundary and hyperbolic interior having the following properties. Given any positive integer n, there exist n distinct slopes $\alpha_1, ..., \alpha_n$ and infinitely many pairwise non-commensurable closed, orientable, immersed, incompressible surfaces $F \looparrowright M$, with no incompressible annulus joining F and ∂M, such that F compresses in $M(\alpha_i)$ for all $i = 1, ..., n$.*

The manifold in Theorem 1.1 is M_8, the exterior of the figure eight knot in S^3. Our proof involves a careful analysis of a construction of closed, immersed, totally geodesic surfaces in M_8 which compress in $M_8(\gamma)$ for some

specific γ. In particular, we inspect the proof of the following theorem from [6].

Theorem 1.2. *Suppose* $4 \mid p$ *and* $3 \nmid p$. *Then for any* q *that is relatively prime to* p *there exists infinitely many non-commensurable, closed, immersed, totally geodesic surfaces in* M_8 *which compress in* $M_8(\frac{p}{q})$.

This paper is organized as follows: Section 2 contains a few definitions and constructions from 3-dimensional topology necessary for our work. We then give a brief review of some basic definitions and facts concerning hyperbolic 3-manifolds in Section 3. Section 4 contains the various constructions of totally geodesic surfaces and theorems on compressing totally geodesic surfaces. Section 5 includes definitions and facts from number theory and quadratic forms, and we prove the main technical theorem needed for the proof of Theorem 1.1. In Section 6, we prove Theorem 1.1.

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2. 3-dimensional Topology

In this section we recall some definitions and facts from 3-dimensional topology. For more details, see [2, 13, 15].

Let M be a compact, orientable 3-manifold with a torus boundary $\partial M \cong T^2$ and let $\pi_1(\partial M) \cong \pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ be generated by λ and μ. A *slope* on ∂M is an isotopy class of simple closed curves on ∂M, and can be uniquely associated (up to inverses) with a primitive element $\alpha = \lambda^p \mu^q \in \pi_1(\partial M)$. Primitivity implies p, q are relatively prime and so the set slopes are in a one-to-one correspondence with $\mathbb{Q} \bigcup \{\infty\}$, where $\lambda^p \mu^q$ corresponds to $\frac{p}{q}$ in the lowest terms. We write $\alpha = \frac{p}{q}$ (with $\infty = \frac{1}{0}$). If $\alpha = \frac{p}{q}$ and $\sigma = \frac{r}{s}$ are two slopes in $\pi_1(\partial M)$, then the *distance* between α and σ is given by $\Delta(\alpha, \sigma) = |ps - qr|$.

Now let α be a slope on ∂M, $S^1 \times D^2$ be a solid torus and $\mu_0 = \{*\} \times \partial D^2$ be a meridional curve on $\partial(S^1 \times D^2)$. We form a closed 3-manifold by α-*Dehn filling* on ∂M by attaching $S^1 \times D^2$ to M identifying $\partial(S^1 \times D^2)$ with ∂M so that α is identified with μ_0. The resulting space, denoted by $M(\alpha)$, is a closed 3-manifold depending only on α up to homeomorphism.

Let F be a closed, connected, orientable surface which is not homeomorphic to a 2-sphere. We say that an immersion $f : F \longrightarrow M$ is an *incompressible surface* if the induced map $f_* : \pi_1(F) \longrightarrow \pi_1(M)$ is injective, and *compressible*, otherwise. A surface $f : F \longrightarrow M$ is *essential* if it is incompressible and is not homotopic into ∂M. We say $f : F \longrightarrow M$ is

acylindrical if no element of $f_*(\pi_1(F))$ is *peripheral*, that is, conjugate into $\pi_1(\partial M)$. Equivalently, there is no annulus in M joining a non-trivial loop in F to a loop in ∂M.

The manifold M we are interested in here will be the interior of compact manifold $\overline{M}$ with torus boundary. We will write $M(\alpha)$ for $\overline{M}(\alpha)$, and will refer to ∂M for $\partial\overline{M}$. We will generally not distinguish between M and $\overline{M}$ when no confusion arises.

The figure eight knot $K \subset S^3$ is the knot whose projection is shown in Figure 1. The manifold we will analyze is $M_8 = S^3 - K$ which is the interior of a compact manifold with torus boundary. In the next two sections, we will describe this manifold in more detail.

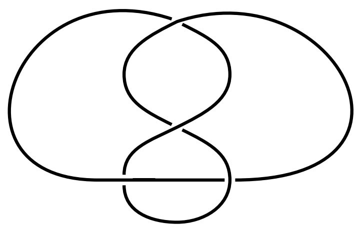

FIGURE 1. The figure eight knot.

3. Hyperbolic 3-manifolds

Here we review some of the background concerning hyperbolic 3-manifolds. See [9, 10] for more details.

Let us consider the upper half space

$$\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$$

endowed with the complete Riemannian metric

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2},$$

which is a model for hyperbolic 3-space. The *boundary at infinity* $\partial\mathbb{H}^3$ is $\widehat{\mathbb{C}} = (\mathbb{C} \times \{0\}) \bigcup \{\infty\}$. The group of all orientation-preserving isometries of $\mathbb{H}^3$ is isomorphic to $PSL_2(\mathbb{C})$ acting by conformal extension of Möbius transformation on $\widehat{\mathbb{C}}$.

As a convention, since $PSL_2(\mathbb{C}) \cong SL_2(\mathbb{C})/\pm I$, whenever we refer to a matrix for an element in $PSL_2(\mathbb{C})$, we really mean one of the corresponding matrices in $SL_2(\mathbb{C})$ under the quotient homomorphism. A subgroup Γ of $PSL_2(\mathbb{C})$ is said to be a *Kleinian group* if the induced topology on Γ is the discrete topology. Equivalently, Γ acts properly discontinuously on $\mathbb{H}^3$.

Throughout this paper, we will consider Γ a torsion-free Kleinian group. Let $M_\Gamma = \mathbb{H}^3/\Gamma$ be the quotient *hyperbolic 3-manifold* with its induced

metric, so we have $\pi_1(M_\Gamma) \cong \Gamma$. We say that Γ is *co-compact* or has *finite co-volume* if M_Γ is compact or has finite total volume, respectively.

As is shown in [12] (see also [9, 10]), $M_8 \cong \mathbb{H}^3/\Gamma_8$, and $\pi_1(M_8) \cong \Gamma_8$, where

$$\Gamma_8 = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \right\rangle,$$

with $\omega = \frac{-1+\sqrt{-3}}{2}$. We note $\mathbb{Z}[\omega] = \mathcal{O}_3$, the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{-3})$. Furthermore Γ_8 has index 12 in $PSL_2(\mathcal{O}_3)$.

4. Surfaces in hyperbolic 3-manifolds

In this section we collect some of definitions and facts concerning surface in hyperbolic 3-manifolds. See [7, 9, 15] for more details.

4.1. Totally geodesic surfaces in $M_\Gamma = \mathbb{H}^3/\Gamma$.

All closed, orientable, immersed, totally geodesic surfaces in $M_\Gamma = \mathbb{H}^3/\Gamma$ arise as follows.

Let $\mathcal{C}$ be any circle in $\widehat{\mathbb{C}}$, that is, a circle or line in $\mathbb{C}$. For any subgroup $\Gamma \subseteq PSL_2(\mathbb{C})$, define

$$Stab_\Gamma(\mathcal{C}) = \{g \in \Gamma \mid g(\mathcal{C}) = \mathcal{C} \text{ and } g \text{ preserves the components of } \widehat{\mathbb{C}}\backslash\mathcal{C}\}.$$

For any circle $\mathcal{C}$, a discrete subgroup of $Stab_{PSL_2(\mathbb{C})}(\mathcal{C})$ is called a *Fuchsian group.* Because of the transitive action of $PSL_2(\mathbb{C})$ on circles in $\widehat{\mathbb{C}}$, there exists $g \in PSL_2(\mathbb{C})$ such that $g(\mathcal{C}) = \widehat{\mathbb{R}} = \mathbb{R}\bigcup\{\infty\}$, and hence $gStab_{PSL_2(\mathbb{C})}(\mathcal{C})g^{-1} = PSL_2(\mathbb{R})$.

Any circle $\mathcal{C}$ in $\widehat{\mathbb{C}}$ bounds a hyperbolic plane $P_\mathcal{C} \cong \mathbb{H}^2$ embedded totally geodesically in $\mathbb{H}^3$. If $\Gamma^{'} = Stab_\Gamma(\mathcal{C})$ is a torsion free Fuchsian group, we obtain a *hyperbolic surface* $S_{\Gamma'} = P_\mathcal{C}/\Gamma^{'}$ with $\pi_1(S_{\Gamma'}) \cong \Gamma^{'}$.

Let Γ be a finite co-volume torsion free Kleinian group such that there exists a circle $\mathcal{C} \subset \widehat{\mathbb{C}}$ for which $\Gamma^{'} = Stab_\Gamma(\mathcal{C})$ has finite co-area. One can check that this induces a proper totally geodesic incompressible immersion

$$S_{\Gamma'} \cong P_\mathcal{C}/\Gamma^{'} \looparrowright M_\Gamma \cong \mathbb{H}^3/\Gamma.$$

Let us write $\mathcal{C}_D$ to denote a circle centered at the origin with radius $D \in \mathbb{Z}^+$. Consider the subgroup of Γ_8,

$$\Gamma_D = Stab_{\Gamma_8}(\mathcal{C}_D) = \{\gamma \in \Gamma_8 \mid \gamma(\mathcal{C}_D) = \mathcal{C}_D\},$$

which always has finite co-area [8].

We say Γ_D, $\Gamma_{D'}$ are *commensurable* if there exists an element $g \in \Gamma_8$ such that $(g\Gamma_D g^{-1})\bigcap\Gamma_{D'}$ is a finite index subgroup in both $g\Gamma_D g^{-1}$ and $\Gamma_{D'}$. The following is a consequence of arithmeticity (see [6, 11]).

Theorem 4.1. *For each positive integer $D \equiv 2 \pmod 3$, Γ_D is a co-compact Fuchsian group and therefore $S_D = P_D/\Gamma_D \looparrowright M_8$ is a closed totally geodesic surface, which is in particular acylindrical. Moreover, Γ_D and $\Gamma_{D'}$ are commensurable in Γ_8 if and only if $D = D^{'}$.*

4.2. **Compressing totally geodesic surfaces.** For the proof of the main theorem, we analyze the construction used in the proof of Theorem 1.2. The details of its proof are in [6].

The construction is started by considering $\pi_1(\partial \overline{M}_8) \cong \mathbb{Z} \oplus \mathbb{Z}$ which is generated by the standard meridian-longitude μ and λ.

For any given integers p and q with $\gcd(p,q)=1$, set $\sigma = \lambda^p \mu^q \in \pi_1(\partial \overline{M}_8)$. Van Kampen's theorem implies that

$$\pi_1\left(M_8\left(\frac{p}{q}\right)\right) = \Gamma_8/\langle\langle\sigma\rangle\rangle$$

where $\langle\langle\sigma\rangle\rangle$ is the normal closure of $\{\sigma\}$ in Γ_8.

Now suppose $4 \mid p$, $3 \nmid p$. For any positive integer k, construct integers n_k, D_k as follows:

$$\begin{array}{rclcl} n_k & = & n_k(p,q) & = & -3(p^2+12q^2)(2+3k)+9 \\ D_k & = & D_k(p,q) & = & (p^2+12q^2)(n_k)^2+2+3k. \end{array}$$

Define a sequence $\{\Gamma_{D_k}\}_{k=1}^{\infty}$ of pairwise non-commensurable, co-compact Fuchsian subgroups $\Gamma_{D_k} = Stab_{\Gamma_8}(\mathcal{C}_{D_k})$ of Γ_8. From Theorem 4.1 we obtain the sequence $\{S_{D_k} = P_{D_k}/\Gamma_{D_k} \looparrowright M_8\}_{k=1}^{\infty}$ of pairwise non-commensurable, closed, orientable, immersed, totally geodesic surfaces in M_8 with $\pi_1(S_{D_k}) \cong \Gamma_{D_k}$.

We can now restate a more precise version of Theorem 1.2, the main theorem of [6].

Theorem 4.2. *For any integer p such that $4 \mid p$, $3 \nmid p$ and q relatively prime to p, let $\{D_k\}_{k=1}^{\infty} = \{D_k(p,q)\}_{k=1}^{\infty}$ be as above. Then for every k, the closed, immersed, totally geodesic surface $S_{D_k} \looparrowright M_8$ compresses in $M_8(\frac{p}{q})$.*

We note that $D_k(p,q)$ depends only on k and p^2+12q^2. Our approach to prove the main theorem is to show that for a given integer n, we can find at least n ways to represent the form p^2+12q^2, where p,q satisfy the above hypothesis. More precisely, there exists a family $\{(p_i,q_i)\}_{i=1}^{m}$, where $m \geq n$, such that $D_k(p_i,q_i) = D_k(p_j,q_j)$ for all $i,j = 1,...,m$ and all positive integer k. By Theorem 4.2, there are infinitely many closed, immersed, totally geodesic surfaces $S_{D_k} \looparrowright M_8$ which compress in $M_8(\frac{p_i}{q_i})$ for all $i = 1,...,m$ and all positive integer k. To find such representations $\{(p_i,q_i)\}_{i=1}^{m}$, we will need some facts about quadratic forms.

5. Quadratic forms

In this section we recall the relevant facts from number theory and basic properties of Legendre symbol and quadratic forms that will be important tools for the proof of the main theorem; (see [3, 5] for more details).

For any integer a and positive odd prime p, the *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } p \mid a \\ +1, & \text{if } p \nmid a \text{ and there exists an integer } x \text{ such that } x^2 \equiv a \pmod{p} \\ -1, & \text{otherwise.} \end{cases}$$

We list here some well-known properties of the Legendre symbol we will need.

Proposition 5.1. *Let p, q be distinct, positive, odd primes, and a, b be integers,*

(1) *(Completely multiplicative law)* $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

(2) *(Quadratic reciprocity law)*

If $p \equiv 1 \pmod 4$ *or* $q \equiv 1 \pmod 4$ *, then* $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)$.

If $p \equiv q \equiv 3 \pmod 4$*, then* $\left(\frac{p}{q}\right) = (-1)\left(\frac{q}{p}\right)$.

(3) *(First supplement to the quadratic reciprocity law)*

$\left(\frac{-1}{p}\right) = 1$ *if and only if* $p \equiv 1 \pmod 4$.

Using this, we prove the following.

Lemma 5.2. *For any prime p greater than 3, $\left(\frac{3}{p}\right) = 1$ if and only if $p \equiv 1$ or $11 \pmod{12}$.*

Proof. Any prime p greater than 3 has $p \equiv 1, 5, 7$ or $11 \pmod{12}$. Given such p apply Proposition 5.1 part (2), and the fact that for any integer x, $x^2 \equiv 0$ or $1 \pmod 3$. □

A quadratic form $f(x, y) = ax^2 + bxy + cy^2$ is called *primitive* if its coefficients, a, b and c are relatively prime. We say an integer m is *represented* by $f(x, y)$ if the equation $m = f(x, y)$ has an integer solution. If this solution has x and y relatively prime, then we say that m is *properly represented* by (x, y). We declare two primitive forms $f(x, y)$ and $g(x, y)$ to be *properly equivalent*, and write $f(x, y) \sim g(x, y)$, if there exist integers p, q, r and s such that $f(x, y) = g(px + qy, rx + sy)$ and $ps - rq = 1$. One can easily check that this defines an equivalence relation.

The *discriminant* of the form $f(x, y) = ax^2 + bxy + cy^2$ is $\mathcal{D} = b^2 - 4ac$. Direct computation shows that if $f(x, y) = g(px + qy, rx + sy)$, then $\mathcal{D}_f = (ps - qr)^2 \mathcal{D}_g$, where $\mathcal{D}_f$ and $\mathcal{D}_g$ are the discriminants of the forms f and g, respectively. This implies that properly equivalent forms have the same discriminant. We restrict our discussion only to the case $\mathcal{D} < 0$, and then f is *positive definite*. A primitive positive definite form $f(x, y) = ax^2 + bxy + cy^2$ is said to be a *reduced form* if $|b| \leq a \leq c$, and if $|b| = a$ or $a = c$ then $b \geq 0$.

Each equivalence class has a good representative quadratic form by Lagrange's Theorem of Reduced Forms.

Theorem 5.3. *Every primitive positive definite form is properly equivalent to a unique reduced form.*

Note that for a fixed discriminant $\mathcal{D} < 0$, there are only finitely many reduced forms. Therefore, the number of classes of primitive, positive definite forms of discriminant $\mathcal{D}$ is finite. To see this, consider a reduced form $f(x,y) = ax^2 + bxy + cy^2$. By definition of a reduced form, we have $b^2 \le a^2, a \le c$. This implies

$$-\mathcal{D} = 4ac - b^2 \ge 4a^2 - a^2 = 3a^2,$$

and

$$0 < a \le \sqrt{\frac{-\mathcal{D}}{3}}, \text{ then } |b| \le a \le \sqrt{\frac{-\mathcal{D}}{3}}\ .$$

Hence there are only finitely many of choices for the integers a, b and c. For example we have the following.

Lemma 5.4. *There are exactly two reduced forms of discriminant $\mathcal{D} = -48$; namely, $3x^2 + 4y^2$ and $x^2 + 12y^2$.*

Proof. From the discussion above, a reduced form $f(x,y) = ax^2 + bxy + cy^2$ of discriminant $\mathcal{D} = -48$ must satisfies $|b| \le a \le \sqrt{\frac{48}{3}} = 4$, and $0 \le c \le 16$. An explicit finite search reveal that $3x^2 + 4y^2$ and $x^2 + 12y^2$ are the only posibilities. $\square$

We will need the following theorem of Gauss.

Theorem 5.5. *Let m be a positive odd number relatively prime to $k > 1$. Then the number of ways that m is properly represented by a reduced form of discriminant $-4k$ is*

$$2\prod_{p|m}\left(1 + \left(\frac{-k}{p}\right)\right),$$

where the product is over all distinct positive prime divisors p of m.

This theorem allows us to prove the following.

Theorem 5.6. *For a given positive integer $m \equiv 7 \pmod{12}$ with all prime divisors congruent to 1 or 7 $\pmod{12}$, the number of proper representations of m by the primitive positive form $3x^2 + 4y^2$ is $2^{\tau(m)+1}$ where $\tau(m)$ is the number of positive prime divisors of m.*

Proof. Let us first investigate the properties of m. Observe that $m \equiv 7 \pmod{12}$ implies that m is odd and $m \equiv 3 \pmod 4$. Since 3 is not a square $\pmod 4$, m cannot be properly represented by the form $x^2 + 12y^2$.

With the given conditions on the divisors of m, we can write

$$m = {p_1}^{\alpha_1} \cdots {p_s}^{\alpha_s} {q_1}^{\beta_1} \cdots {q_t}^{\beta_t}$$

where $p_1, ..., p_s$ are distinct positive primes congruent to 1 $\pmod{12}$ and $q_1, ..., q_t$ are distinct positive primes congruent to 7 $\pmod{12}$. Then

$$7 \equiv m \equiv 7^{\beta_1 + \cdots + \beta_t} \pmod{12}$$

and it follows that

$$\beta_1 + \cdots + \beta_t \equiv 1 \pmod 2.$$

Now we consider the proper equivalence classes of the fixed discriminant -48. By Lemma 5.4, there are exactly 2 classes of primitive, positive definite reduced forms $3x^2 + 4y^2$ and $x^2 + 12y^2$. However, as noted above, m cannot be represented by the latter form. Therefore, appealing to Theorem 5.5, Proposition 5.1 and Lemma 5.2, the number of ways that m is properly represented by a reduced form $3x^2 + 4y^2$ is

$$\begin{aligned}
2\prod_{p|m}\left(1+\left(\frac{-12}{p}\right)\right) &= 2\prod_{p|m}\left(1+\left(\frac{-3}{p}\right)\left(\frac{4}{p}\right)\right) \\
&= 2\prod_{i=1}^{s}\left(1+\left(\frac{-3}{p_i}\right)\right)\prod_{i=1}^{t}\left(1+\left(\frac{-3}{q_i}\right)\right) \\
&= 2\prod_{i=1}^{s}\left(1+\left(\frac{-1}{p_i}\right)\left(\frac{3}{p_i}\right)\right)\prod_{i=1}^{t}\left(1+\left(\frac{-1}{q_i}\right)\left(\frac{3}{q_i}\right)\right) \\
&= 2\prod_{i=1}^{s}(1+(1)(1))\prod_{i=1}^{t}(1+(-1)(-1)) \\
&= 2^{(s+t)+1} = 2^{\tau(m)+1}.
\end{aligned}$$

□

As a consequence, we have the following corollary.

Corollary 5.7. *Let $N = 4m$ for some positive integer m such that $m \equiv 7 \pmod{12}$ and all prime divisors of m are congruent to 1 or 7 $\pmod{12}$. The number of ways to properly represent N in the form $N = p^2 + 12q^2$, where $4 \mid p$ and $3 \nmid p$, is exactly $2^{\tau(N)}$.*

Proof. Writing $p = 4r$ sets up to bijection between the proper representations of $N = p^2 + 12q^2$ and $m = 3q^2 + 4r^2$. So it suffices to prove that the number of ways to properly represent $m = 3q^2 + 4r^2$ is $2^{\tau(N)} = 2^{\tau(m)+1}$.

Applying Theorem 5.6, the number of ways to represent $m = 3q^2 + 4r^2$ is exactly $2^{\tau(m)+1}$, as required. □

6. Compression

In this section we prove

Theorem 6.1. *Given any positive integer n, there exist n distinct slopes $\alpha_1, ..., \alpha_n$ in the ∂M_8 and infinitely many closed, orientable, immersed, incompressible surfaces $S_{D_k} \looparrowright M_8$ with no incompressible annulus joining S_{D_k} and ∂M_8 which compress in $M_8(\alpha_i)$ for all $i = 1, ..., n$ and positive integer k.*

From this, Theorem 1.1 easily holds.

Proof of Theorem 1.1. By assuming Theorem 6.1, this theorem immediately follows when we let $M = \overline{M}_8$ which is a compact, orientable, irreducible 3-manifold with torus boundary. We note that $M(\alpha) = M_8(\alpha)$ for any slope α in ∂M.

□

To prove Theorem 6.1 we first recall Dirichlet's Theorem on arithmetic progressions (see [14] for more details).

Theorem 6.2. *If positive numbers s and t are relatively prime, then there are infinitely many primes p such that $p \equiv s \pmod{t}$.*

Using this, we can prove

Lemma 6.3. *For any given positive integer n, there exists a family of n pairs $\{(p_i, q_i)\}_{i=1}^n$ such that $p_i^2 + 12q_i^2 = p_j^2 + 12q_j^2$ for all $i, j = 1, ..., n$, where p_i and q_i are relatively prime, $4 \mid p_i$, but $3 \nmid p_i$ for all $i = 1, ..., n$.*

Proof. For any given positive integer n, there exists a positive integer k such that $n \leq 2^k$. Define the integer N by

$$N = 4{a_1}^{\beta_1} \cdots {a_{k-1}}^{\beta_{k-1}},$$

where $a_1, ..., a_{k-1}$ are distinct positive primes congruent to $7 \pmod{12}$ and $\beta_1 + \cdots + \beta_{k-1} \equiv 1 \pmod 2$. Since 7 and 12 are relatively prime, we know that such integer N exists by Dirichlet's theorem.

By construction, N satisfies the hypothesis of Collorary 5.7. Therefore, there exist $2^{\tau(N)} = 2^k \geq n$ pairs (p, q) which properly represent N, and moreover these satisfy the conditions of the lemma.

□

Example 6.4. *Using Mathematica for $n = 16$, we have the family*
$\{(32, 813), (200, 811), (680, 789), (1112, 747), (1328, 717), (1528, 683), (1640, 661), (1912, 597), (2032, 563), (2320, 461), (2560, 339), (2608, 307), (2648, 277), (2720, 211), (2752, 173), (2792, 107)\}$.

Each such pairs (p, q) are relatively prime, $4 \mid p$, $3 \nmid p$ and $p^2 + 12q^2 = 7,932,652$.

Proof of Theorem 6.1. For any given n, by Lemma 6.3 there exists a family $\{(p_i, q_i)\}_{i=1}^n$ such that $p_i^2 + 12q_i^2 = p_j^2 + 12q_j^2$ for all $i, j = 1, ..., n$, where p_i and q_i are relatively prime such that $4 \mid p_i$, but $3 \nmid p_i$ for all $i = 1, ..., n$. For each $i = 1, ..., n$, consider the slope $\alpha_i = \frac{p_i}{q_i}$ on ∂M_8. As noted in Section 4.2, $D_k(p_i, q_i) = D_k(p_j, q_j)$ for all $i, j = 1, ..., n$ and all $k > 0$, and we denote this simply as D_k. Theorem 4.2 implies S_{D_k} compresses in $M_8(\alpha_i)$ for all $i = 1, ..., n$ and all $k > 0$. Since $D_1 < D_2 < ...$, Theorem 4.1 implies S_{D_k} are all non-commensurable.

□

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